Some Examples of Comparisons of Connecting Networks

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In the theory of telephone traffic it is of interest to compare the performance of connecting networks, as measured by the probability by blocking, when they are subjected to the same traffic sources. The question arises whether there are examples of pairs of networks, with the same number of crosspoints, whose respective graphs of loss as a function of offered load cross each other. The existence of such examples would establish the principle that some network configurations are inherently more efficient at some traffic levels than at others, so that the "excellence" of a network is not necessarily a purely combinatorial notion independent of offered traffic. Examples of the above phenomenon are exhibited which do not involve only very small networks.

I. INTRODUCTION

In the theory of telephone traffic it is of interest to compare the performance of connecting networks, as measured by the probability of blocking, when they are subjected to the same traffic sources. Naturally, there are cases in which the result of this comparison is independent of the calling rate λ.[1] In this connection, H. O. Pollak has raised the question whether there are examples of pairs of networks, with the same number of crosspoints, the first of which is better than the second at one value of λ, while the second is better than the first at another value of λ.

The existence of such examples would establish the principle that some network configurations (in particular, some switch sizes) are inherently more efficient at some traffic levels than at others, so that the "excellence" of a network is not necessarily a purely combinatorial notion independent of offered traffic. We shall exhibit examples of the above phenomenon which are nontrivial in that they do not involve only very small networks.

II. PRELIMINARIES

The notations and conventions of Refs. 2 and 3 will be used. We shall need machinery for studying the probability of blocking at very high values of the traffic λ; this is provided by the natural expansion of the equilibrium state probabilities in inverse powers of λ:

Lemma: The state probabilities $\{p_x, x \in S\}$ *can be expanded in a power series*

$$p_x = \sum_{m=0}^{\infty} d_m(x)\lambda^{-m} \tag{1}$$

valid for λ *real and sufficiently large. With* $w = \max_{x \in S} |x|$, *the coefficients* $d_m(x)$ *have the property*

$$d_m(x) = 0 \quad \text{for} \quad 0 \leqq m < w - |x|, \tag{2}$$

and the numbers $d_{w-|x|}(x)$ satisfy

$$\begin{aligned} \sum_{|x|=w} d_0(x) &= 1 \\ s(y)d_{w-|y|}(y) &= \sum_{z \in A_y} d_{w-|z|}(z), \qquad |y| < w, \end{aligned} \tag{3}$$

$$d_{w-|x|}(x) \geqq 0. \tag{4}$$

Proof: $p_x(\lambda)$ is a rational function of λ, and so has an expansion of the form (1) if λ is large enough. Substitution of (1) into the equilibrium condition gives these equations for the coefficients $d(\cdot)$: (No unblocked call is rejected.)

$$|x|\, d_{m-1}(x) + s(x)d_m(x) = \sum_{y \in A_x} d_{m-1}(y) + \sum_{y \in B_x} d_m(y)r_{yx}.$$

It follows at once that if 0 = zero state (with no calls up), then $d_0(0) = 0$, and

$$s(x)d_0(x) = \sum_{y \in B_x} d_0(y)r_{yx},$$

so that $d_0(x) = 0$ unless x is maximal in the natural partial ordering of states.

Thus, if x is not maximal then

$$d_m(x) = 0 \quad \text{for} \quad |x| < w - m$$

holds for $m = 0$. Assume that it holds for some $m - 1 \geqq 0$. For x not maximal, $s(x) > 0$ and

$$s(x)d_m(x) = -|x|\,d_{m-1}(x) + \sum_{y \epsilon A_x} d_{m-1}(y) + \sum_{y \epsilon B_x} d_m(y)r_{yx}.$$

If $|x| < w - m$, then $d_{m-1}(x) = 0$ and $y \epsilon A_x$ implies $d_{m-1}(y) = 0$, both by the induction hypothesis. Thus, $d_m(x)$ is expressible as a constant times $d_m(0)$. But

$$s(0)d_m(0) = \sum_{|y|=1} d_{m-1}(y) = 0,$$

by the induction hypothesis.

If x is maximal with $|x| < w - m$, then

$$|x|\,d_m(x) = \sum_{y \epsilon B_x} d_{m+1}(y)r_{yx}, \qquad m \geqq 0.$$

But $y \epsilon B_x$ implies $|y| = |x| - 1 < w - m - 1$, and so $d_{m+1}(y) = 0$. (1) and (2) imply (3) and (4).

The formula

$$\sum_{|x|=w} d_0(x) = 1,$$

follows from

$$p_x = \sum_{m=w-|x|} d_m(x)\lambda^{-m}$$

and $\sum_{x \epsilon S} p_x = 1$ by letting $\lambda \to \infty$.

It follows from the lemma just proved that for sufficiently high values of the traffic parameter λ, the probability of blocking has the form

$$\Pr\{\mathrm{bl}\} = \frac{\sum_{|x|=k} \beta_x d_{w-k}(x)}{\sum_{|x|\geq k} \alpha_x \sum_{j=w-|x|}^{w-k} d_j(x)\lambda^{w-k-j}} + o(1), \qquad \lambda \to \infty,$$

where k is the greatest integer such that some states with k calls in progress have blocked calls ($\beta_x > 0$). In particular, we see that

$$\lim_{\lambda\to\infty} \Pr\{\mathrm{bl}\} = \begin{cases} 0 & \text{if} \quad k < w \\ 1 & \text{if} \quad k = w. \end{cases}$$

III. COMPARISONS

The examples to be studied are the networks A and B in Figs. 1 and 2, respectively. Both are three-stage networks of the type due to C. Clos,[4] each with nr inlets (outlets). We show that there are values of

m, n, and r such that (*i*) A and B have very nearly the same number of crosspoints, and (*ii*) A has lower blocking than B at all sufficiently low values of the traffic λ, while B has lower blocking than A at all sufficiently high values of λ. Tbe calculations forming this comparison will be carried out in the traffic model of Chapter 8 of Ref. 2; familiarity with this model is assumed.

In A, at least m calls must be in progress in order for there to be any blocking. Hence,[5]

$$\Pr\{\mathrm{bl}\}_A = \kappa\lambda^m + o(\lambda^m), \qquad \lambda \to 0.$$

In B there is a least integer $k \geqq 0$ such that

$$\Pr\{\mathrm{bl}\}_B = c\lambda^k + o(\lambda^k), \qquad \lambda \to 0$$

with $c > 0$. We shall show that $k \leqq r + 1$, independently of the routing used to run B. It has been shown[5] that the probability p_x of a state in the model of Ref. 2 is of the form

$$p_x = p_0 \sum_\pi \lambda^{\frac{1}{2}[l(\pi)+|x|]} \prod_{y\in\pi} \frac{1}{|y| + \lambda s(y)}, \tag{5}$$

where the sum is over paths π on $(S, \leqq)$ permitted by the routing rule in use starting at 0 and ending at x, the product is along the path, and $l(\pi)$ is the path-length.

In B it takes r calls in progress to block a call. Choose an outer switch on each side of B and consider a sequence of r attempted calls, each of wbich is from one of these switches to tbe otber, together with one

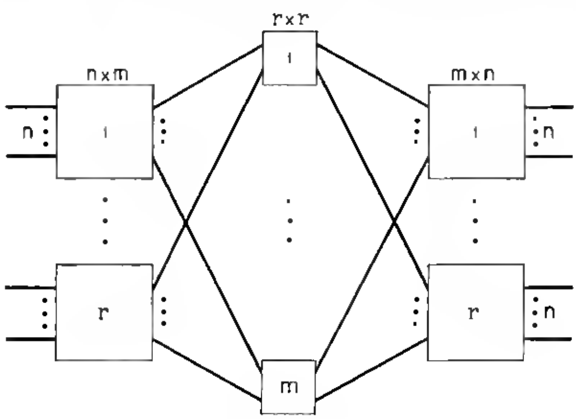

Fig. 1 — Network A: $2mnr + mr^2$ crosspoints.

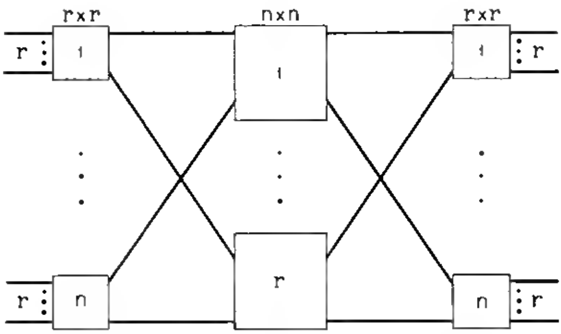

Fig. 2 — Network B: $2nr^2 + rn^2$ crosspoints.

more call c. The last call, c, will have to go on one of the r middle switches each of which already has exactly one call. (Fig. 3, upper half.) If now the *other call* on the switch carrying c hangs up (Fig. 3, lower half), we will have reached a blocking state from 0 with positive probability along a path π of length $l(\pi) = r + 2$. Since the blocking state reached has r calls in progress, there is a contribution in formula (5) of the form

$$c\lambda^{r+1}, \qquad c > 0.$$

It follows that if $m > r + 1$, then

$$\Pr\{\mathrm{bl}\}_A < \Pr\{\mathrm{bl}\}_B$$

for all λ sufficiently small.

Now take $n > m$, so that Lemma 1 gives

$$\Pr\{\mathrm{bl}\}_A = \frac{\sum_{|x|=nr} \beta_x d_0(x)}{\sum_{|x|=nr} \alpha_x d_0(x)} + o(1)$$

$$= 1 + o(1), \qquad \lambda \to \infty.$$

At the same time, it can be seen that in network B, $\beta_x = 0$ for $|x| > nr - 2$, so that

$$\Pr\{\mathrm{bl}\}_B = \frac{\sum_{|x|=nr-2} \beta_x d_2(x)}{\sum_{|x|\geq nr-2} \alpha_x \sum_{j=nr-|x|}^{2} d_j(x)\lambda^{2-j}} + o(1).$$

For $|x| = nr$,

$$| x | \, d_0(x) = \sum_{y \epsilon B_x} d_1(y) r_{yx} .$$

Thus,

$$nr = \sum_{|x|=nr} | x | \, d_0(x) = \sum_{|x|=nr} \sum_{y \epsilon S} d_1(y) r_{yx} = \sum_{|y|=nr-1} d_1(y) s(y)$$
$$= \sum_{|y|=nr-1} d_1(y),$$

and since $\alpha_x = 0$ for $| x | = nr$, and $\alpha_y = s(y) = 1$ for $| y | = nr - 1$.

$$\Pr\{\text{bl}\}_B = \frac{\displaystyle\sum_{|z|=nr-2} \beta_z d_2(z)}{\displaystyle\sum_{|x|=nr-2} \alpha_x d_2(x) + \lambda \sum_{|y|=nr-1} d_1(y)} + o(1) \cdot$$

The leading term is <1, and so for all λ sufficiently large

$$\Pr\{\text{bl}\}_B < \Pr\{\text{bl}\}_A .$$

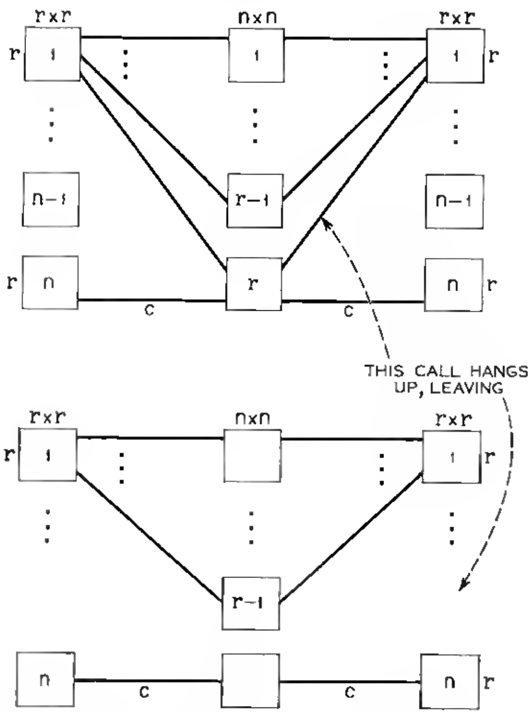

Fig. 3 — Blocking state of A reached in $r + 2$ steps (only links in use indicated).

It remains to show that there are values of m, n, r such that $n > m > r + 1$ for which the number of crosspoints of A is very nearly equal to that of B. Picking $m = r + 2$, the condition for equality is that

$$n = 2 + (4 + 2r + r^2)^{\frac{1}{2}}.$$

With $[t]$ the integer part of t, we pick n as

$$2 + [(4 + 2r + r^2)^{\frac{1}{2}}] > 2 + (1 + 2r + r^2)^{\frac{1}{2}} = r + 3 > m.$$

With this choice of n A actually has more crosspoints than B and yet gives higher blocking at large values of λ than B does.

REFERENCES

1. Beneš, V. E., On Comparing Connecting Networks, to appear in J. Combinatorial Theor.
2. Beneš, V. E., *Mathematical Theory of Connecting Networks and Telephone Traffic*, Academic Press, New York, 1965.
3. Beneš, V. E., Programming and Control Problems Arising from Optimal Routing in Telephone Networks, abstract in SIAM J. Control, *4*, February, 1966, pp. 6–18; B.S.T.J., *45*, November, 1966, pp. 1373–1438.
4. See Chapter 3 of Ref. 2.
5. See Chapter 8 of Ref. 2.